Problem Set 5 due April 8, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Invent an algorithm (explain all the steps in words, and explain why it works) which takes general vectors $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ as inputs, and decides whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. *(15 points)*

Solution: To decide whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$, you need to check two things:

- $\boldsymbol{p}$ lies in the plane spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, i.e. there exist numbers α and β such that:

  $$\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{1}$$

  Equating coefficients implies:

  $$\begin{cases} \alpha + \beta = p_1 \\ 2\alpha - \beta = p_2 \\ 3\alpha = p_3 \end{cases} \tag{2}$$

  and solving for α and β via back substitution using the last two equations gives us:

  $$\alpha = \frac{p_3}{3} \quad \text{and} \quad \beta = \frac{2p_3}{3} - p_2$$

  Now plugging these formulas in the first equation of (2) gives us:

  $$\boxed{p_3 - p_2 = p_1}$$

  which is precisely the condition for $\boldsymbol{p}$ to lie in the plane spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

- $\boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. This means one needs to check that:

$$1 \cdot (b_1 - p_1) + 2 \cdot (b_2 - p_2) + 3 \cdot (b_3 - p_3) = 0$$

  and:

$$1 \cdot (b_1 - p_1) + (-1) \cdot (b_2 - p_2) = 0$$

Note: we will also accept any other explicit algorithm, including a computation of the matrix:

$$A(A^T A)^{-1} A^T$$

as an explicit 3×3 matrix, and the answer is something along the lines of:

$$\textit{check that } \boldsymbol{p} = (\textit{an explicit } 3 \times 3 \textit{ matrix}) \cdot \boldsymbol{b}$$

But we would expect a complete computation of the explicit matrix $A(A^T A)^{-1} A^T$.

Grading Rubric: 9 points for the first bullet, 6 points for the second bullet. For each of these, half of the points will be taken off if the correct condition is stated without proper explanation, and 1-2 points will be taken off for minor computation errors.

Problem 2:

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Let $V = N(A)$ be the nullspace of A, and $W = C(A^T)$ be the row space of A.

(1) Compute bases of V and W. *(10 points)*

(2) Compute the projection matrices P_V onto V and P_W onto W. *(10 points)*

(3) Compute $P_V + P_W$. The answer should be a very nice matrix. Explain geometrically why you get this answer (*Hint: it has to do with the geometric relation between V and W*). *(5 points)*

Solution: (1) The reduced row echelon form of this matrix is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the row space W is spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. As for the nullspace V, it consists of vectors:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \Leftrightarrow \quad x + 2y + 3z = 0$$

The pivot variable is x and the free variables are y and z. Therefore, V has two basis vectors, and for the former we set $y = 1$, $z = 0$ while for the latter we set $y = 0$, $z = 1$ and then we solve for x. We conclude that a basis of V is given by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Grading Rubric (for each of the 3 basis vectors):

- Correct answer with complete explanation (it can be argued differently from the above, e.g. direct computation and back subsitution) *(5 points)*
- Minor computation errors *(3-4 points)*
- Correct answer but inadequate justification *(2 points)*
- Missing or very incorrect answer *(0 points)*

(2) We need to set up matrices X and Y whose column spaces are V and W, respectively. To this end, we can take:

$$X = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Then the projection matrices aee given by the formulas:

$$P_V = X(X^T X)^{-1} X^T = \frac{1}{14} \begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$$

$$P_W = Y(Y^T Y)^{-1} Y^T = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Grading Rubric (for each of the 2 matrices):

- Correct answer, from which the formula $P_V = X(X^T X)^{-1} X^T$ is obvious. *(5 points)*
- Correct method, but minor computation error *(4 points)*
- Correct answer but formula for P_V missing *(2 points)*
- Just constructed the matrices X and Y, but went no further *(1 point)*
- Missing or very incorrect answer *(0 points)*

(3) It's easy to see that $P_V + P_W = I$. This is actually true for all pairs of orthogonally complementary vector spaces V and W (such as the nullspace and row space of a matrix, in the case at hand). The reason is that any vector $\boldsymbol{a} \in \mathbb{R}^3$ can be uniquely written as the sum of its projections onto orthogonal complements:

$$\boldsymbol{a} = \text{proj}_V \boldsymbol{a} + \text{proj}_W \boldsymbol{a} \quad \Rightarrow \quad \boldsymbol{a} = P_V \boldsymbol{a} + P_W \boldsymbol{a} = (P_V + P_W)\boldsymbol{a}$$

Since this must hold for all vectors $\boldsymbol{a}$, we need $P_V + P_W = I$.

Grading Rubric:

- Correct answer and justification (students may argue algebraically, geometrically, or just show a picture, but in the latter case it should be manifest that the reason is that vectors decompose as the sums of their projections onto the two subspaces) *(5 points)*
- Just the answer $P_V + P_W = I$, but no pertinent explanation *(2 points)*
- Incorrect answer *(0 points)*

Problem 3:

Consider the following lines L_1 and L_2 in 3-dimensional space:

$$L_1 = \left\{ \begin{bmatrix} x+1 \\ x \\ x \end{bmatrix} \text{ for } x \in \mathbb{R} \right\} \qquad \text{and} \qquad L_2 = \left\{ \begin{bmatrix} y \\ 2y \\ 3y \end{bmatrix} \text{ for } y \in \mathbb{R} \right\}$$

(1) Which of these is a subspace and which is not? Explain why. *(5 points)*

(2) Consider a point $R = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ on the line which **IS NOT** a subspace among L_1 and L_2 (so if R is on L_1, the entries a, b, c are given in terms of x, while if R is on L_2, they are given in terms of y). Compute the smallest possible distance from R to the other line among L_1 and L_2. *(5 points)*

(3) By minimizing the quantity in part (2), find the points $P \in L_1$ and $Q \in L_2$ for which the distance $|PQ|$ is minimal among all possible choices of a point on either line. *(5 points)*

(4) What can you say about the line PQ in relation to the lines L_1 and L_2? *(5 points)*

Solution: (1) L_1 is not a subspace, because it does not contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. On the other hand, L_2 is a subspace because it is the column space of the matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Grading rubric: Full points only with correct justification (although such justification may differ from the one above): 3 points for L_1 and 2 points for L_2.

(2) Fix a number x, and consider the point $P = \begin{bmatrix} x+1 \\ x \\ x \end{bmatrix}$ in L_1. We seek to determine the distance from P to L_2. The closest point $Q \in L_2$ to the given point P is simply its projection onto L_2, i.e.:

$$Q = \text{proj}_{L_2} P$$

Since the line L_2 is spanned by the vector $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we have:

$$Q = \boldsymbol{a}\frac{\boldsymbol{a} \cdot P}{\boldsymbol{a} \cdot \boldsymbol{a}} = \boldsymbol{a}\frac{x+1+2x+3x}{14} = \boldsymbol{a}\frac{6x+1}{14} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \frac{6x+1}{14}$$

The distance between P and Q is then given by:

$$|PQ| = \sqrt{\left(x+1-\frac{6x+1}{14}\right)^2 + \left(x - \frac{12x+2}{14}\right)^2 + \left(x - \frac{18x+3}{14}\right)^2} =$$

$$= \sqrt{\frac{(8x+13)^2 + (2x-2)^2 + (-4x-3)^2}{14}} = \sqrt{\frac{3x^2}{7} + \frac{8x}{7} + \frac{13}{14}}$$

Grading Rubric:

- Correct answer and justification *(5 points)*
- Noted that we need $Q = \text{proj}_{L_2} P$, and had the correct formula for the projection *(3 points)*
- Noted that we need $Q = \text{proj}_{L_2} P$, but incorrect formula for the projection *(2 points)*
- Missing or very incorrect answer *(0 points)*

(3) The quantity $\frac{3x^2}{7} + \frac{8x}{7} + \frac{13}{14}$ is minimized when $x = -\frac{4}{3}$, which corresponds to:

$$P = \frac{1}{3}\begin{bmatrix} -1 \\ -4 \\ -4 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

Grading Rubric:

- Correct answer and justification *(5 points)*
- Noted that we need to minimize $\frac{3x^2}{7} + \frac{8x}{7} + \frac{13}{14}$ (or whatever other formula they got in the previous part, to account for algebra errors) *(2 points)*
- Missing or very incorrect answer *(0 points)*

(4) The line PQ is perpendicular to both L_1 and L_2. Indeed, the direction of this line is the vector:

$$\frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

which is perpendicular to the directions of the lines L_1 and L_2, i.e. the vectors:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Grading rubric: 5 points for noting that $PQ \perp L_1, L_2$ (no justification needed).

Problem 4:

Consider the following cubic curve in the xy plane: $y = ax^3 + bx + c$.

(1) Compute a, b, c such that the curve passes through the points $(0,1)$, $(1,0)$ and $(2,5)$ (don't just guess, use linear algebra to solve for a, b, c). *(10 points)*

(2) Compute a, b, c such that the cubic curve is the best fit for the points $(0,1)$, $(1,0)$, $(2,5)$ and $(3,-1)$: this means that the sum of the squares of the vertical distances between the curve and the four given points should be minimum (*Hint: this is done similarly to the example of fitting a line, that we did at the end of Lecture 13*). *(10 points)*

Solution: (1) The three points $(0,1)$, $(1,0)$, $(2,5)$ lying on the curve is equivalent to the following system of equations:

$$\begin{cases} c = 1 \\ a + b + c = 0 \\ 8a + 2b + c = 5 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 8 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

To solve this system, let's put the matrix in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 8 & 2 & 1 & 5 \end{array}\right] \xrightarrow{\text{switch rows 1 and 3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 8 & 2 & 1 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_2 - 8r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -6 & -7 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_2 \cdot (-\frac{1}{6})}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{7}{6} & -\frac{5}{6} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{6} & \frac{5}{6} \\ 0 & 1 & \frac{7}{6} & -\frac{5}{6} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_2 - \frac{7}{6}r_3} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{6} & \frac{5}{6} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 + \frac{r_3}{6}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, the system is equivalent to $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, so the curve is $y = x^3 - 2x + 1$.

Grading Rubric:

- Correct answer and process *(10 points)*
- Minor computational errors, but correct process *(8-9 points)*
- Missing or very incorrect answer *(0 points)*

(2) The four points $(0,1)$, $(1,0)$, $(2,5)$, $(3,-1)$ lying on the curve is equivalent to the following system of equations:

$$\begin{cases} c = 1 \\ a + b + c = 0 \\ 8a + 2b + c = 5 \\ 27a + 3b + c = -1 \end{cases} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 8 & 2 & 1 \\ 27 & 3 & 1 \end{bmatrix}}_{\text{call this } A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\text{call this } \boldsymbol{v}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 5 \\ -1 \end{bmatrix}}_{\text{call this } \boldsymbol{b}}$$

The system of equations $A\boldsymbol{v} = \boldsymbol{b}$ has no solutions on the nose, but out goal is to find an "almost" solution. According to the problem statement, this entails finding a numbers a, b, c (i.e. the vector $\boldsymbol{v}$) such that the quantity:

$$(c-1)^2 + (a+b+c-0)^2 + (8a+2b+c-5)^2 + (27a+3b+c+1)^2 = ||A\boldsymbol{v} - \boldsymbol{b}||^2$$

is minimal. We know that this is achieved for that $\boldsymbol{v}$ such that:

$$A\boldsymbol{v} = \text{proj}_{C(A)}\boldsymbol{b} = A(A^TA)^{-1}A^T\boldsymbol{b}$$

and the natural candidate for such a $\boldsymbol{v}$ is:

$$\boldsymbol{v} = (A^TA)^{-1}A^T\boldsymbol{b} = \begin{bmatrix} -\frac{1}{3} \\ \frac{17}{6} \\ 0 \end{bmatrix}$$

(we expect students to compute the inverse of the 3×3 matrix A^TA using Gauss-Jordan elimination, although we do allow them to use a calculator for the basic addition/multiplication/simplification of fractions that arise at each step of the algorithm; however, we do want to see all the steps in Gauss-Jordan elimination explicitly laid out). Therefore, the best fit curve is $y = -\frac{x^3}{3} + \frac{17x}{6}$.

Problem 5:

(1) Use Gram-Schmidt to compute an orthonormal basis of $\mathbb{R}^4$ that includes the vector:

$$\boldsymbol{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(10 points)

(2) Compute the $A = QR$ factorization of the matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -5 & -1 \\ 2 & -5 & -3 \end{bmatrix}$$

(where Q has orthonormal columns and R is square upper triangular). *(10 points)*

Solution: (1) Let's apply Gram-Schmidt to the basis $\boldsymbol{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^4$ (to see that the above forms a basis, note that $\boldsymbol{q}_1$ cannot be a linear combination of $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$, because the latter three vectors have a zero top entry, and $\boldsymbol{q}_1$ does not). First we modify $\boldsymbol{v}_2$:

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \text{proj}_{\boldsymbol{q}_1}\boldsymbol{v}_2 = \boldsymbol{v}_2 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

The vector $\boldsymbol{w}_2$ has length $\sqrt{\frac{3}{4}}$, so we need to scale it to have unit length:

$$\boldsymbol{q}_2 = \frac{\boldsymbol{w}_2}{\sqrt{3/4}} = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

Then we modify $\boldsymbol{v}_3$:

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \text{proj}_{\boldsymbol{q}_1}\boldsymbol{v}_3 - \text{proj}_{\boldsymbol{q}_2}\boldsymbol{v}_3 = \boldsymbol{v}_3 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_3) - \boldsymbol{q}_2(\boldsymbol{q}_2 \cdot \boldsymbol{v}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

The vector $\boldsymbol{w}_3$ has length $\sqrt{\frac{2}{3}}$, so we need to scale it to have unit length:

$$\boldsymbol{q}_3 = \frac{\boldsymbol{w}_3}{\sqrt{2/3}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Finally, we modify $\boldsymbol{v}_4$:

$$\boldsymbol{w}_4 = \boldsymbol{v}_4 - \text{proj}_{\boldsymbol{q}_1}\boldsymbol{v}_4 - \text{proj}_{\boldsymbol{q}_2}\boldsymbol{v}_4 - \text{proj}_{\boldsymbol{q}_3}\boldsymbol{v}_4 = \boldsymbol{v}_4 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_4) - \boldsymbol{q}_2(\boldsymbol{q}_2 \cdot \boldsymbol{v}_4) - \boldsymbol{q}_3(\boldsymbol{q}_3 \cdot \boldsymbol{v}_4) =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The vector $\boldsymbol{w}_4$ has length $\sqrt{\frac{1}{2}}$, so we need to scale it to have unit length:

$$\boldsymbol{q}_4 = \frac{\boldsymbol{w}_4}{\sqrt{1/2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$ form an orthonormal basis.

Grading Rubric:

- Did 3/3 steps correctly *(10 points)*
- Did 2/3 steps correctly *(8 points)*
- Did 1/3 steps correctly *(5 points)*
- Incorrect method, or did not choose a basis to start with *(0 points)*

(2) As we have seen in class, coming up with the QR factorization of a matrix is the same thing as doing Gram-Schmidt on its columns, and keeping track of which column operations we do at each step. So let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ be the columns of A:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -5 \\ -5 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$$

Start from $\boldsymbol{w}_1 = \boldsymbol{v}_1$. The first step in Gram-Schmidt is to rescale $\boldsymbol{w}_1$ so that it has length 1:

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{3} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The second step in Gram-Schmidt is to modify the vector $\boldsymbol{v}_2$:

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_2) = \boldsymbol{v}_2 + 6\boldsymbol{q}_1 = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

The third step in Gram-Schmidt is to rescale $\boldsymbol{w}_2$ so that it has length 1:

$$\boldsymbol{q}_2 = \frac{\boldsymbol{w}_2}{\sqrt{18}} = \frac{1}{\sqrt{18}}\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

The fourth step in Gram-Schmidt is to modify the vector $\boldsymbol{v}_3$:

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_3) - \boldsymbol{q}_2(\boldsymbol{q}_2 \cdot \boldsymbol{v}_3) = \boldsymbol{v}_3 + 3\boldsymbol{q}_1 + 0\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The fifth step in Gram-Schmidt is to rescale $\boldsymbol{w}_3$ so that it has length 1:

$$\boldsymbol{q}_3 = \frac{\boldsymbol{w}_3}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ thus computed are the columns of the Q matrix:

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{-1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{\sqrt{18}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

To get the R matrix, we need to interpret all the five column operations above as multiplication by appropriate elimination and diagonal matrices:

$$AD_1^{\left(\frac{1}{3}\right)} E_{12}^{(6)} D_2^{\left(\frac{1}{\sqrt{18}}\right)} E_{13}^{(3)} E_{23}^{(0)} D_3^{\left(\frac{1}{\sqrt{2}}\right)} = Q$$

Let us move all the elimination and diagonal matrices to the right-hand side:

$$A = Q \underbrace{D_3^{(\sqrt{2})} E_{23}^{(0)} E_{13}^{(-3)} D_2^{(\sqrt{18})} E_{12}^{(-6)} D_1^{(3)}}_{\text{call this matrix } R}$$

It's easy to explicitly compute $R = \begin{bmatrix} 3 & -6 & -3 \\ 0 & 3\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

- Correct answer and argument with all the steps *(10 points)*
- Minor computation errors *(8-9 points)*
- Correct answer but missing key steps *(5 points)*
- Did Gram-Schmidt on the columns of A but did not get the QR factorization *(5 points)*
- No answer or incorrect method *(0 points)*